

On the positive definiteness and eigenvalues of meet and join matrices

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MIKA MATTILA* AND PENTTI HAUKKANEN
School of Information Sciences
FI-33014 University of Tampere, Finland

Abstract

In this paper we study the positive definiteness of meet and join matrices using a novel approach. When the set S_n is meet closed, we give a sufficient and necessary condition for the positive definiteness of the matrix $(S_n)_f$. From this condition we obtain some sufficient conditions for positive definiteness as corollaries. We also use graph theory and show that by making some graph theoretic assumptions on the set S_n we are able to reduce the assumptions on the function f while still preserving the positive definiteness of the matrix $(S_n)_f$. Dual theorems of these results for join matrices are also presented. As examples we consider the so-called power GCD and power LCM matrices as well as MIN and MAX matrices. Finally we give bounds for the eigenvalues of meet and join matrices in cases when the function f possesses certain monotonic behaviour.

Key words and phrases:

Meet matrix, Join matrix, GCD matrix, LCM matrix, Smith determinant

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* *Corresponding author.* Tel.: +358 50 318 5881, fax: +358 3 219 1001

E-mail addresses: mika.mattila@uta.fi (M. Mattila),
pentti.haukkanen@uta.fi (P. Haukkanen)

1 Introduction

The research of GCD and LCM matrices was initiated by H. J. S. Smith [33] in 1875 when he studied the determinant of the $n \times n$ matrix in which the ij element is the greatest common divisor (i, j) of i and j . He also considered the $n \times n$ matrix with the least common multiple $[i, j]$ of i and j as its ij element. During the next century the determinants of GCD type matrices were a topic of interest for many number theorists and linear algebraists (see the references in [18]; the two articles [29] and [35] by Lindström and Wilf are especially relevant). In 1989 Beslin and Ligh [11] initiated a new wave of more intense research of GCD matrices, which soon led to poset-theoretic generalizations of GCD matrices. Rajarama Bhat [32] gave the definition of meet matrix, and Haukkanen [14] was the first to study these matrices systematically. Join matrices were defined later by Korkee and Haukkanen [27].

Over the years many authors have considered the positive definiteness of GCD, LCM, meet and join matrices. In 1989 Beslin and Ligh [9] showed that the GCD matrix (S) of the set $S = \{x_1, \dots, x_n\}$, in which the ij element is (x_i, x_j) , is positive definite. Four years later Bourque and Ligh [10] proved that if f is an arithmetical function such that

$$d \mid x_i \text{ for some } x_i \in S \Rightarrow (f * \mu)(d) > 0,$$

then the GCD matrix $(S)_f$ with $f((x_i, x_j))$ as its ij element is positive definite. In [11] Beslin and Ligh report results concerning the positive definiteness of GCD matrices with respect to generalized Ramanujan's sums. In 2001 Korkee and Haukkanen [26] gave a sufficient condition for positive definiteness of meet matrices, and in [27] they presented a similar condition for join matrices. A couple of years later Altinisik et al. [4] obtained a sufficient and necessary condition for positive definiteness of a matrix closely related to meet matrices. At the same time Ovall [31] went back to GCD and LCM matrices and showed that GCD and certain reciprocal matrices are positive definite, whereas some reciprocal matrices and certain LCM matrices are indefinite. In 2006 Bhatia [12] showed once again that the usual GCD matrix is infinitely divisible and therefore positive definite. Later Bhatia [13] also studied certain MIN matrices and presented six proofs for their positive definiteness (it should be noted that MIN matrices can easily be seen as special cases of meet matrices).

There are also some results for the eigenvalues of GCD-type matrices to be found in the literature. Wintner [36] published results concerning the largest eigenvalue of the $n \times n$ matrix having

$$\left(\frac{(i, j)}{[i, j]} \right)^\alpha$$

as its ij entry and subsequently Lindqvist and Seip [28] investigated the asymptotic behaviour of the smallest and largest eigenvalue of the same matrix. More recently Hilberdink [19] and also Berkes and Weber [8] addressed this same topic from an analytical perspective. The first paper concerning the eigenvalues of proper GCD matrices was by Balatoni [7] as he considered the eigenvalues of the classical Smith's GCD matrix.

One way to obtain information about the eigenvalues of GCD type matrices is to study the norms of these matrices. The O estimates of the norms have been studied in many papers, see [2, 6, 15, 16, 17]. Hong and Loewy [21, 22] studied the asymptotic behaviour of a special kind of GCD matrices, Altinisik [3] provides information about the eigenvalues of GCD matrices, a paper by Hong and Enoch Lee [20] addresses the eigenvalues of reciprocal LCM matrices and there is also one paper about the eigenvalues of meet and join matrices by Ilmonen et al. [24].

In this paper we provide new information about the positive definiteness and the eigenvalues of meet and join matrices. The notations and most of the concepts are defined in Section 2. Section 3 contains some new characterizations and key examples of positive definite meet and join matrices. In Section 4 we make use of graph theory and study the positive definiteness of meet and join matrices from this new graph theoretic perspective. In Section 5 we provide upper bounds for all the eigenvalues of meet and join matrices in which the function f evinces certain monotonic behaviour.

2 Preliminaries

Throughout this paper (P, \preceq) is an infinite but locally finite lattice, $f : P \rightarrow \mathbb{R}$ is a real-valued function on P and $(x_n)_{n=1}^\infty$ is an infinite sequence of distinct elements of P such that

$$x_i \preceq x_j \Rightarrow i \leq j. \quad (2.1)$$

For every $n \in \mathbb{Z}_+$, let $S_n = \{x_1, x_2, \dots, x_n\}$. The set S_n is said to be *meet closed* if $x_i \wedge x_j \in S_n$ for all $x_i, x_j \in S_n$, in other words, the structure (S_n, \preceq) is a meet semilattice. The concept of *join closed set* is defined dually.

The $n \times n$ matrix having $f(x_i \wedge x_j)$ as its ij element is the *meet matrix* of the set S_n with respect to f and is denoted by $(S_n)_f$. Similarly, the $n \times n$ matrix having $f(x_i \vee x_j)$ as its ij element is the *join matrix* of the set S_n with respect to f and is denoted by $[S_n]_f$. When $(P, \preceq) = (\mathbb{Z}_+, |)$ the matrices $(S_n)_f$ and $[S_n]_f$ are referred to as the GCD and LCM matrices of the set S_n with respect to f .

Let $D_n = \{d_1, d_2, \dots, d_{m_n}\}$ be any finite subset of P containing all the elements $x_i \wedge x_j$, where $x_i, x_j \in S_n$, and having its elements arranged so that

$$d_i \preceq d_j \Rightarrow i \leq j.$$

Next we define the function $\Psi_{D_n, f}$ on D_n inductively as

$$\Psi_{D_n, f}(d_k) = f(d_k) - \sum_{d_v \prec d_k} \Psi_{D_n, f}(d_v), \quad (2.2)$$

or equivalently

$$f(d_k) = \sum_{d_v \preceq d_k} \Psi_{D_n, f}(d_v). \quad (2.3)$$

Thus we have

$$\Psi_{D_n, f}(d_k) = \sum_{d_v \preceq d_k} f(d_v) \mu_{D_n}(d_v, d_k), \quad (2.4)$$

where μ_{D_n} is the Möbius function of the poset (D_n, \preceq) , see [1, Section IV.1] and [34, Proposition 3.7.1].

Let E_{D_n} denote the $n \times m_n$ matrix defined as

$$(e_{D_n})_{ij} = \begin{cases} 1 & \text{if } d_j \preceq x_i, \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

The matrix E_{D_n} may be referred to as the incidence matrix of the set D_n with respect to the set S_n and the partial ordering \preceq .

The set D_n , the function $\Psi_{D_n, f}$ and the matrix E_{D_n} are needed when considering the matrix $(S_n)_f$. Next we define the dual concepts which we use in the study of the matrix $[S_n]_f$.

Let $B_n = \{b_1, b_2, \dots, b_{l_n}\}$ be any finite subset of P containing all the elements $x_i \vee x_j$ with $x_i, x_j \in S_n$ and having its elements arranged so that

$$b_i \preceq b_j \Rightarrow i \leq j.$$

We define the function $\Phi_{B_n, f}$ on B_n inductively as

$$\Phi_{B_n, f}(b_k) = f(b_k) - \sum_{b_v \prec b_k} \Phi_{B_n, f}(b_v), \quad (2.6)$$

or equivalently

$$f(b_k) = \sum_{b_v \preceq b_k} \Phi_{B_n, f}(b_v). \quad (2.7)$$

Thus we have

$$\Phi_{B_n, f}(b_k) = \sum_{b_v \preceq b_k} f(b_v) \mu_{B_n}(b_k, b_v), \quad (2.8)$$

where μ_{B_n} is the Möbius function of the poset (B_n, \preceq) .

Let E_{B_n} denote the $n \times l_n$ matrix defined as

$$(e_{B_n})_{ij} = \begin{cases} 1 & \text{if } b_j \succeq x_i, \\ 0 & \text{otherwise.} \end{cases} \quad (2.9)$$

We refer to the matrix E_{B_n} as the incidence matrix of the set B_n with respect to the set S_n and the partial ordering \succeq .

Remark 2.1. If we are only interested in the positive definiteness and eigenvalues of meet and join matrices, then the condition (2.1) is, in fact, not necessary but can still be made without restricting generality. If S_n does not satisfy the condition (2.1) and S'_n is a set obtained from S_n by rearranging its elements so that (2.1) holds, then there exists a permutation matrix P such that

$$(S'_n)_f = P(S_n)_f P^T = P(S_n)_f P^{-1}.$$

Thus the matrices $(S'_n)_f$ and $(S_n)_f$ are similar and therefore have the same eigenvalues, positive definiteness properties etc.

It is well known (see, for example [5, 30]) that adopting the above notations the matrices $(S_n)_f$ and $[S_n]_f$ can be factored as

$$(S_n)_f = E_{D_n} \Lambda_{D_n,f} E_{D_n}^T \quad \text{and} \quad [S_n]_f = E_{B_n} \Delta_{B_n,f} E_{B_n}^T, \quad (2.10)$$

where

$$\Lambda_{D_n,f} = \text{diag}(\Psi_{D_n,f}(d_1), \Psi_{D_n,f}(d_2), \dots, \Psi_{D_n,f}(d_{m_n}))$$

and

$$\Delta_{B_n,f} = \text{diag}(\Phi_{B_n,f}(b_1), \Phi_{B_n,f}(b_2), \dots, \Phi_{B_n,f}(b_{l_n})).$$

By using the first factorization in a case when the set S_n is meet closed, it is easy to show (see [5, Theorem 4.2]) that

$$\det(S_n)_f = \Psi_{S_n,f}(x_1) \Psi_{S_n,f}(x_2) \cdots \Psi_{S_n,f}(x_n). \quad (2.11)$$

Similarly, when the set S_n is join closed we have

$$\det[S_n]_f = \Phi_{S_n,f}(x_1) \Phi_{S_n,f}(x_2) \cdots \Phi_{S_n,f}(x_n) \quad (2.12)$$

(see [30, Theorem 4.2]). In the next section these determinant formulas appear also to be useful when considering the positive definiteness of meet and join matrices.

3 On the positive definiteness of meet and join matrices

We begin our study by considering the positive definiteness of the matrix $(S_n)_f$ in case when the set S_n is meet closed. Under these circumstances we are able to give sufficient and necessary conditions for positive definiteness of the matrix $(S_n)_f$. Theorem 3.1 is also closely related to Theorem 5.1 in [4].

Theorem 3.1. *If the set S_n is meet closed, then the matrix $(S_n)_f$ is positive definite if and only if $\Psi_{S_n,f}(x_i) > 0$ for all $i = 1, 2, \dots, n$.*

Proof. Since removing a maximal element does not affect the meet closeness of the set S_i , it follows that all the sets $S_n, S_{n-1}, \dots, S_2, S_1$ are meet closed. In addition, the determinants of the matrices $(S_i)_f$, where $i = 1, 2, \dots, n$, are the leading principal minors of the matrix $(S_n)_f$. By (2.11) we have

$$\begin{aligned} \det(S_1)_f &= \Psi_{S_n,f}(x_1) \\ \det(S_2)_f &= \Psi_{S_n,f}(x_1)\Psi_{S_n,f}(x_2) \\ &\vdots \\ \det(S_{n-1})_f &= \Psi_{S_n,f}(x_1)\Psi_{S_n,f}(x_2) \cdots \Psi_{S_n,f}(x_{n-1}) \\ \det(S_n)_f &= \Psi_{S_n,f}(x_1)\Psi_{S_n,f}(x_1) \cdots \Psi_{S_n,f}(x_{n-1})\Psi_{S_n,f}(x_n). \end{aligned}$$

Now $(S_n)_f$ is positive definite if and only if $\det(S_i)_f > 0$ for all $i = 1, 2, \dots, n$ (see [23, Theorem 7.2.5]), and the determinants above are all positive if and only if $\Psi_{S_n,f}(x_i) > 0$ for all $i = 1, 2, \dots, n$. \square

Next we present a dual theorem for join matrices.

Theorem 3.2. *If the set S_n is join closed, then the matrix $[S_n]_f$ is positive definite if and only if $\Phi_{S_n,f}(x_i) > 0$ for all $i = 1, 2, \dots, n$.*

Proof. Let us denote

$$S'_1 = \{x_n\}, S'_2 = \{x_{n-1}, x_n\}, \dots, S'_{n-1} = \{x_2, \dots, x_{n-1}, x_n\}.$$

Since the determinants of the matrices

$$[S'_1]_f, [S'_2]_f, \dots, [S'_{n-1}]_f \text{ and } [S_n]_f$$

constitute a nested sequence of n principal minors of $[S_n]_f$, the matrix $[S_n]_f$ is positive definite if and only if all of these matrices have positive determinants (again, see [23, Theorem 7.2.5]). Since all these sets are join closed, the determinants can be calculated by using (2.12). The rest of the proof is similar to the proof of Theorem 3.1. \square

Example 3.1. Let S_n be a chain. Thus $x_1 \prec x_2 \prec \dots \prec x_{n-1} \prec x_n$. Clearly, the set S_n is both meet and join closed (the matrices $(S_n)_f$ and $[S_n]_f$ may be referred to as the MIN and MAX matrices of the chain S_n respectively). In this case we have $\Psi_{S_n,f}(x_1) = f(x_1)$ and

$$\Psi_{S_n,f}(x_i) = \sum_{x_k \preceq x_i} f(x_k) \mu_{S_n}(x_k, x_i) = f(x_i) - f(x_{i-1})$$

for all $i = 2, \dots, n$. Now it follows from Theorem 3.1 that the matrix $(S_n)_f$ is positive definite if and only if $f(x_1) > 0$ and $f(x_i) > f(x_{i-1})$ for all $i = 2, \dots, n$. In other words, we must have

$$0 < f(x_1) < f(x_2) < \dots < f(x_{n-1}) < f(x_n).$$

If we set $(P, \preceq) = (\mathbb{Z}^+, \leq)$, $f(k) = k$ for all $k \in \mathbb{Z}$ and $S_n = \{1, 2, \dots, n\}$, we obtain the MIN matrix studied recently by Bhatia [13]. Among other things, he presents six distinct proofs for the positive definiteness of this matrix. The one in this example is yet another different proof.

Similarly, by using Theorem 3.2 it is possible to show that the matrix $[S_n]_f$ is positive definite if and only if

$$0 < f(x_n) < f(x_{n-1}) < \dots < f(x_2) < f(x_1).$$

Next we focus on the case when the set S_n is neither meet nor join closed. It turns out that by using Theorems 3.1 and 3.2 it is possible to say something about the positive definiteness of the matrices $(S_n)_f$ and $[S_n]_f$ also under these circumstances. Corollary 3.1 may be seen as a generalization of Theorem 1 (i) in [11].

Corollary 3.1. *Let D_n be any finite meet closed subset of P containing all the elements of S_n . If $\Psi_{D_n, f}(d_i) > 0$ for all $d_i \in D_n$, then the matrix $(S_n)_f$ is positive definite.*

Proof. By Theorem 3.1 the matrix $(D_n)_f$ is positive definite. Thus the matrix $(S_n)_f$ is also positive definite since it is a principal submatrix of the matrix $(D_n)_f$. \square

Example 3.2. Let $(P, \preceq) = (\mathbb{Z}^+, |)$, $\alpha \in \mathbb{R}$ and $f(n) = n^\alpha$ for all $n \in \mathbb{Z}^+$. Under these assumptions the matrices $(S_n)_f$ and $[S_n]_f$ become the so-called power GCD and LCM matrices, which have been studied extensively by Hong et al. [20, 21]. It is well known that the matrix $(S_n)_f$ is positive definite if $\alpha > 0$ (see [10, Example 1] and [11, Example 3]). Here we give another proof for this by using the previous corollary.

Let

$$D_n = \downarrow S_n = \{d \in \mathbb{Z}^+ \mid d \mid x_i \text{ for some } x_i \in S_n\}.$$

Let $*$ denote the Dirichlet convolution and μ denote the number-theoretic Möbius function. Now for every $d_k \in D_n$ we have

$$\Psi_{D_n, f}(d_k) = \sum_{d_v \mid d_k} d_v^\alpha \mu\left(\frac{d_k}{d_v}\right) = (f * \mu)(d_k) = J_\alpha(d_k) = d_k^\alpha \prod_{p \mid d_k} \left(1 - \frac{1}{p^\alpha}\right),$$

where J_α is a generalization of the Jordan totient function. If $\alpha > 0$, then clearly $J_\alpha(d_k) > 0$ for all $d_k \in D_n$ and therefore by Corollary 3.1 the matrix $(S_n)_f$ is positive definite.

Next we present a similar corollary that concerns the matrix $[S_n]_f$. The proof is essentially the same as the proof of Corollary 3.1.

Corollary 3.2. *Let B_n be any finite join closed subset of P containing all the elements of S_n . If $\Phi_{B_n, f}(b_i) > 0$ for all $b_i \in B_n$, then the matrix $[S_n]_f$ is positive definite.*

Example 3.3. Let $(P, \preceq) = (\mathbb{Z}^+, |)$ as in Example 3.2, $\alpha \in \mathbb{R}^+$ and $f(n) = \frac{1}{n^\alpha}$ for all $n \in \mathbb{Z}^+$. Hong and Enoch Lee [20, Theorem 2.1] have shown that the matrix $[S_n]_f$ is positive definite. Here we present a different proof for this fact by using Corollary 3.2.

Let $\alpha > 0$, let $\downarrow \text{lcm } S_n$ denote the set of divisors of $\text{lcm } S_n$ and let $\uparrow S_n$ stand for the set $\{k \in \mathbb{Z}^+ \mid x_i | k \text{ for some } i = 1, \dots, n\}$. Now let

$$B_n = \uparrow S_n \cap \downarrow \text{lcm } S_n = \{d \in \mathbb{Z}^+ \mid x_i | d \text{ for some } x_i \in S_n \text{ and } d | \text{lcm } S_n\}.$$

Then for every $b_k \in B_n$ we have

$$\begin{aligned} \Phi_{B_n, f}(b_k) &= \sum_{b_k | b_v | \text{lcm } S_n} \frac{1}{b_v^\alpha} \mu\left(\frac{b_v}{b_k}\right) \\ &= \left(\frac{1}{\text{lcm } S_n}\right)^\alpha \sum_{b_k | b_v | \text{lcm } S_n} \left(\frac{\text{lcm } S_n}{b_v}\right)^\alpha \mu\left(\frac{b_v}{b_k}\right) \\ &= \left(\frac{1}{\text{lcm } S_n}\right)^\alpha \sum_{a | \frac{\text{lcm } S_n}{b_k}} \left(\frac{(\text{lcm } S_n)/b_k}{a}\right)^\alpha \mu(a) \\ &= \left(\frac{1}{\text{lcm } S_n}\right)^\alpha J_\alpha\left(\frac{\text{lcm } S_n}{b_k}\right) > 0. \end{aligned}$$

Thus by Corollary 3.2 the matrix $[S_n]_f$ is positive definite.

As seen in the above examples, there are two obvious ways to choose the sets D_n and B_n . The first is to take D_n (resp. B_n) to be the meet (resp. join) subsemilattice of P generated by the set S_n . The other is to take $D_n = \downarrow S_n$ and $B_n = \uparrow S_n$ in case when the sets $\downarrow S_n$ and $\uparrow S_n$ are finite, and otherwise take $D_n = \downarrow S_n \cap \uparrow (\wedge S_n)$ and $B_n = \uparrow S_n \cap \downarrow (\vee S_n)$ (here $\vee S_n = x_1 \vee \dots \vee x_n$ and $\wedge S_n = x_1 \wedge \dots \wedge x_n$). Benefits of both choices are explained in [30].

Although Corollaries 3.1 and 3.2 can be used in many cases, their conditions are not necessary for the positive definiteness of the matrices $(S_n)_f$ and $[S_n]_f$ and thus they are not always applicable. The following example illustrates this.

Example 3.4. Let $(P, \preceq) = (\mathbb{Z}^+, |)$, $S_3 = \{6, 10, 15\}$ and

$$\left\{ \begin{array}{l} f(1) = 0 \\ f(2) = -1 \\ f(3) = 3 \\ f(5) = -2 \\ f(6) = 5 \\ f(10) = 2 \\ f(15) = 3. \end{array} \right.$$

Then we obtain the GCD matrix

$$(S_3)_f = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix},$$

which can easily be shown to be positive definite. However, if we choose the elements of D_3 as in Figure 1, direct calculations show that $\Psi_{D_3,f}(d_2) = -1 < 0$ and $\Psi_{D_3,f}(d_4) = -2 < 0$. Thus the meet matrix $(S_3)_f$ is positive definite although some of the values of $\Psi_{D_3,f}$ are negative.

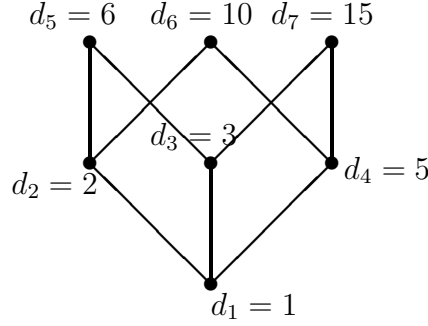


Figure 1: The lattice (D_3, \leq) and the choices of the elements of D_3 .

4 Trees, A -sets and positive definiteness

Next we turn our attention to the special case where the Hasse diagram of the set $\text{meetcl}(S_n)$ is a tree (when it is considered as an undirected graph). Here $\text{meetcl}(S_n)$ (resp. $\text{joincl}(S_n)$) denotes the meet subsemilattice (resp. the join subsemilattice) of P generated by the set S_n . Like in Example 3.1, also in this case a certain monotonicity property of f guarantees the positive definiteness of $(S_n)_f$ (resp. $[S_n]_f$). First we present the definitions of these properties.

Definition 4.1. The set $S_n \subseteq P$ is said to be a \wedge -tree set if the Hasse diagram of $\text{meetcl}(S_n)$ is a tree. Analogously, S_n is a \vee -tree set if the Hasse diagram of $\text{joincl}(S)$ is a tree.

There are also a couple of other characterizations for \wedge -tree sets and \vee -tree sets. We present these only for \wedge -tree sets, since the characterizations for \vee -tree sets are dual to these.

Lemma 4.1. *The following statements are equivalent:*

1. S_n is a \wedge -tree set.

2. Every element in $\text{meetcl}(S_n)$ covers at most one element of $\text{meetcl}(S_n)$.
3. For every $x \in \text{meetcl}(S_n)$ the set

$$(\downarrow x) \cap \text{meetcl}(S_n) = \{y \in \text{meetcl}(S_n) \mid y \preceq x\}$$

is a chain.

4. For all $x, y, z \in \text{meetcl}(S_n)$ we have

$$(x \preceq z \text{ and } y \preceq z) \Rightarrow (x \preceq y \text{ or } y \preceq x).$$

Proof. The proof is simple and straightforward. \square

Next we define the monotonicity property for f that we mentioned earlier.

Definition 4.2. The function $f : P \rightarrow \mathbb{R}$ is *strictly order-preserving* if

$$x \prec y \Rightarrow f(x) < f(y). \quad (4.1)$$

Analogously, f is *strictly order-reversing* if

$$x \prec y \Rightarrow f(y) < f(x). \quad (4.2)$$

The function f is said to be *order-preserving* (resp. *order-reversing*) if equality is allowed on the right side of (4.1) (resp. (4.2)).

Remark 4.1. After adopting the terminology in Definition 4.2 we are able to revisit Example 3.1 and express its results in the following form: If the set S_n is a chain, then

1. $(S_n)_f$ is positive definite
 $\Leftrightarrow f$ is strictly order-preserving in S_n with positive values,
2. $[S_n]_f$ is positive definite
 $\Leftrightarrow f$ is strictly order-reversing in S_n with positive values.

The following theorem presents a condition for positive definiteness of $(S_n)_f$ (resp. $[S_n]_f$) in the case when the values of f are positive and f is order-preserving (resp. order-reversing).

Theorem 4.1. Let $f(x) > 0$ for all $x \in P$. Then the following statements hold:

1. If S_n is a \wedge -tree set and f is strictly order-preserving, then $(S_n)_f$ is positive definite.
2. If S_n is a \vee -tree set and f is strictly order-reversing, then $[S_n]_f$ is positive definite.

Proof. We prove only the first part since the proof of the second part is dual to it. Let $D_n = \text{meetcl}(S_n)$. We apply Corollary 3.1 and show that $\Psi_{D_n, f}(d_k) > 0$ for all $d_k \in D_n$. If $k = 1$, then $d_k = \min D_n$ and we have $\Psi_{D_n, f}(d_k) = f(d_k) > 0$ by assumption. Now let $k > 1$. By Lemma 4.1 d_k covers exactly one element d_w in $\text{meetcl}(S_n)$ and by the order-preserving property we have $f(d_w) < f(d_k)$. Formula (2.3) yields

$$f(d_w) = \sum_{d_v \preceq d_w} \Psi_{D_n, f}(d_v) \text{ and } f(d_k) = \sum_{d_v \preceq d_k} \Psi_{D_n, f}(d_v),$$

and by subtracting we obtain

$$0 < f(d_k) - f(d_w) = \sum_{d_v \preceq d_k} \Psi_{D_n, f}(d_v) - \sum_{d_v \preceq d_w} \Psi_{D_n, f}(d_v) = \Psi_{D_n, f}(d_k),$$

which completes the proof. \square

As seen in Remark 4.1, sometimes it is not only sufficient but also necessary for the positive definiteness of the matrix $(S_n)_f$ that the function f is strictly order-preserving. The next theorem is an example of this. A similar statement can be made regarding join matrices.

Theorem 4.2. *If $x_1 = \min S_n$ and the Hasse diagram of the set S_n is a tree and the matrix $(S_n)_f$ is positive definite, then the function f is strictly order-preserving in S_n and $f(x_i) > 0$ for all $x_i \in S_n$.*

Proof. In this case the set S_n is clearly both meet closed and \wedge -tree set. We begin the proof by showing that if x_j covers x_i , then $f(x_i) < f(x_j)$. By Theorem 3.1 $\Psi_{S_n, f}(x_j) > 0$, and from Equation (2.3) we obtain

$$f(x_j) = \sum_{x_k \preceq x_j} \Psi_{S_n, f}(x_k) \text{ and } f(x_i) = \sum_{x_k \preceq x_i} \Psi_{S_n, f}(x_k).$$

Subtracting the second from the first yields

$$f(x_j) - f(x_i) = \sum_{x_k \preceq x_j} \Psi_{S_n, f}(x_k) - \sum_{x_k \preceq x_i} \Psi_{S_n, f}(x_k) = \Psi_{S_n, f}(x_j) > 0,$$

from which we obtain $f(x_i) < f(x_j)$. Then suppose that $x_i \prec x_j$ but x_j does not cover x_i for some $x_i, x_j \in S_n$. Since (P, \preceq) and in particular (S_n, \preceq) is locally finite, there is only a finite number of elements of S_n in the interval $[x_i, x_j]$. In fact, by item 3 in Lemma 4.1, the elements of the set $S_n \cap [x_i, x_j]$ are always comparable, and therefore the elements of this set form a chain

$$x_i \prec x_{k_1} \prec x_{k_2} \prec \cdots \prec x_{k_r} \prec x_j$$

in which the previous element is always covered by the next. This implies that

$$f(x_i) < f(x_{k_1}) < f(x_{k_2}) < \cdots < f(x_{k_r}) < f(x_j),$$

and thus we have proven the order-preservation of f in S_n in general. The second claim now follows easily. By Theorem 3.1 $f(x_1) = \Psi_{S_n, f}(x_1) > 0$. Further, since $x_1 \preceq x_i$ for all $x_i \in S_n$ and f is strictly order-preserving, $f(x_i) > 0$ for all $x_i \in S_n$. \square

In [25] Korkee studies the meet and join matrices of an A -set, which he defines as follows.

Definition 4.3. The set S_n is an A -set if the set $A = \{x_i \wedge x_j \mid i \neq j\}$ is a chain.

Korkee derives formulas for the structure, determinant and inverse of the matrix $(S_n)_f$ in a case when S_n is an A -set. He also does the same for the matrix $[S_n]_f$ in a case when the dual of S_n is an A -set. He does not, however, consider the positive definiteness of these matrices.

It turns out that Theorem 4.1 can be applied directly to show the positive definiteness of the matrix $(S_n)_f$ when the set S_n is an A -set and the positive definiteness of the matrix $[S_n]_f$ when the dual of the set S_n is an A -set. This follows from the next theorem.

Theorem 4.3. *Every A -set S_n is also a \wedge -tree set and every dual of an A -set is a \vee -tree set.*

Proof. Again we prove only the first part of the claim, since the second part follows from it trivially. Assume that S_n is an A -set. First we need to show that $\text{meetcl}(S_n) = S_n \cup A$, where A is the set defined above. In order to do this, we only need to check that $S_n \cup A$ is meet closed. Let $x, y \in S_n \cup A$. We may assume that $x \in S_n$ and $y \in A$, since the other cases are trivial. Now $y = u \wedge v$ for some $u, v \in S_n$, and we obtain

$$x \wedge y = (x \wedge x) \wedge (u \wedge v) = \underbrace{(x \wedge u)}_{\in A} \wedge \underbrace{(x \wedge v)}_{\in A} \in A,$$

since A is a chain. Thus the first part of the proof is complete.

Next we prove that every element $x \in \text{meetcl}(S_n)$ covers at most one element of $\text{meetcl}(S_n)$. We now suppose for a contradiction that x covers both y and z for some $x, y, z \in \text{meetcl}(S_n)$. Since y and z are incomparable, we must have $y \notin A$ or $z \notin A$ (since A is a chain). We may assume that $z \notin A$, from which it follows that $z \in S_n$. Now we must have $x \notin S_n$, since otherwise we would have $x \wedge z = z \notin A$. Thus $x \in A$ and there exist elements $u, v \in S_n$ such that $x = u \wedge v$. Now, as we can see from Figure 2, we have $v \wedge z = z \notin A$, which is a contradiction. The claim now follows from Lemma 4.1. \square

It is easy to see that the converse of Theorem 4.3 is not true. Figure 3 exemplifies this. It also illustrates the structure of a typical A -set. The Hasse diagram of an A -set is always a tree whether the set S_n is finite or not.

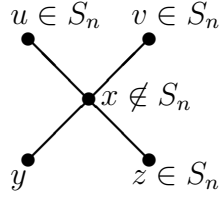


Figure 2: Illustration of the proof of Lemma 4.3.

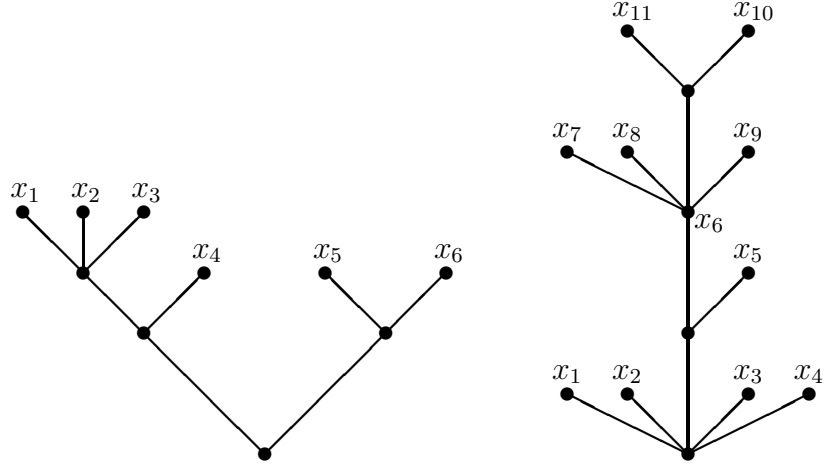


Figure 3: The Hasse diagram on the left is an example of a set S_6 that is a Λ -tree set but not an A -set. The semilattice on the right is an example of a nontrivial finite A -set S_{11} .

5 Eigenvalue estimations

In this section we present bounds for the eigenvalues of certain meet and join matrices. In order to do this, we first need to present the following two lemmas. We here assume that f is strictly order-preserving or order-reversing and also that f is either *increasing* or *decreasing in the set* S_n with respect to the indices i of the elements x_i , i.e. $i \leq j \Rightarrow f(x_i) \leq f(x_j)$ or $i \leq j \Rightarrow f(x_i) \geq f(x_j)$. It should be noted that if $f : P \rightarrow \mathbb{R}$ is either order-preserving or order-reversing, then it is always possible to rearrange the elements of the set S_n so that f becomes increasing or decreasing with respect to the indices. And as stated in Remark 2.1, this does not affect on the eigenvalues. For example, if f is order-preserving, we may list the images of the elements of the set S_n in ascending order as

$$f(x_{j_1}) \leq f(x_{j_2}) \leq \cdots \leq f(x_{j_n}),$$

and then define $x'_i = x_{j_i}$ for all $i = 1, 2, \dots, n$. This even satisfies (2.1), since by order-preserving property we have

$$x'_i \preceq x'_j \Rightarrow f(x'_i) \leq f(x'_j) \Rightarrow i \leq j.$$

Therefore if the function f is order-preserving, assuming $i \leq j \Rightarrow f(x_i) \leq f(x_j)$ causes no additional restrictions in the study of eigenvalues of meet matrices.

Lemma 5.2 and Theorem 5.2 are generalizations of Hong's and Enoch Lee's results (see [20, Theorem 2.3]).

Lemma 5.1. *Let $f : P \rightarrow \mathbb{R}$ be a function with nonnegative values, and let W_k denote the k -dimensional subspace of the complex vector space \mathbb{C}^n consisting of vectors that have zero entries in the coordinates at $k+1, k+2, \dots, n$ (i.e. $W_k = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k\}$). Let $\mathbf{y} = [y_1, \dots, y_n]^T$ be any vector in W_k (that is, $y_{k+1} = \dots = y_n = 0$). If f is order-preserving in $\text{meetcl}(S_n)$, then we have*

$$\mathbf{y}^*(S_n)_f \mathbf{y} \leq k \mathbf{y}^* \mathbf{y} f(x_k), \quad (5.1)$$

where \mathbf{y}^* is the complex conjugate transpose of \mathbf{y} .

Proof. We apply induction on k . In the case when $k = 1$ it is rather trivial that

$$\mathbf{y}^*(S_n)_f \mathbf{y} = \overline{y_1} y_1 f(x_1) = \mathbf{y}^* \mathbf{y} f(x_1),$$

where $\overline{y_1}$ denotes the complex conjugate of y_1 . Our induction hypothesis is that the claim holds for k with $1 \leq k < n$, and next we show that the claim also holds for $k+1$. Let C_i denote the i th column of the matrix $(S_n)_f$, and let $\mathbf{y} \in W_{k+1}$. First we observe that

$$\mathbf{y}^*(S_n)_f \mathbf{y} = \mathbf{y}^* C_1 y_1 + \dots + \mathbf{y}^* C_k y_k + \mathbf{y}^* C_{k+1} y_{k+1}.$$

Now let $\mathbf{z} \in W_k$ such that $z_i = y_i$ for all $i \neq k+1$ and $z_{k+1} = 0$. Thus the quadratic form $\mathbf{z}^*(S_n)_f \mathbf{z}$ is contained in the previous expression and it can be written as

$$\begin{aligned} \mathbf{y}^*(S_n)_f \mathbf{y} &= \mathbf{y}^* C_{k+1} y_{k+1} + \overline{y_{k+1}} f(x_{k+1} \wedge x_1) y_1 + \dots \\ &\quad + \overline{y_{k+1}} f(x_{k+1} \wedge x_k) y_k + \mathbf{z}^*(S_n)_f \mathbf{z}. \end{aligned} \quad (5.2)$$

Next we start to analyse these terms individually. First of all, the order preserving property of f yields that $0 \leq f(x_{k+1} \wedge x_j) \leq f(x_{k+1})$ for all $j = 1, \dots, k$. By also applying the triangle inequality and the simple fact

that $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$ for all $a, b \in \mathbb{C}$ we obtain

$$\begin{aligned}
& |\mathbf{y}^* C_{k+1} \mathbf{y}_{k+1}| \\
&= |y_{k+1}| |\overline{y_1} f(x_{k+1} \wedge x_1) + \cdots + \overline{y_k} f(x_{k+1} \wedge x_k) + \overline{y_{k+1}} f(x_{k+1})| \\
&\leq |y_{k+1}| (|\overline{y_1}| f(x_{k+1} \wedge x_1) + \cdots + |\overline{y_k}| f(x_{k+1} \wedge x_k) + |\overline{y_{k+1}}| f(x_{k+1})) \\
&\leq |y_{k+1}| (|\overline{y_1}| + \cdots + |\overline{y_k}| + |\overline{y_{k+1}}|) f(x_{k+1}) \\
&\leq \left(|y_{k+1}|^2 + \frac{1}{2} \sum_{i=1}^k (|y_{k+1}|^2 + |y_i|^2) \right) f(x_{k+1}) \\
&= \frac{f(x_{k+1})}{2} ((k+1) |y_{k+1}|^2 + \mathbf{y}^* \mathbf{y}). \tag{5.3}
\end{aligned}$$

Very similarly

$$\begin{aligned}
& |\overline{y_{k+1}} f(x_{k+1} \wedge x_1) y_1 + \cdots + \overline{y_{k+1}} f(x_{k+1} \wedge x_k) y_k| \\
&\leq f(x_{k+1}) |y_{k+1}| (|y_1| + \cdots + |y_k|) \\
&\leq \frac{f(x_{k+1})}{2} \sum_{i=1}^k (|y_{k+1}|^2 + |y_i|^2) = \frac{f(x_{k+1})}{2} ((k-1) |y_{k+1}|^2 + \mathbf{y}^* \mathbf{y}). \tag{5.4}
\end{aligned}$$

Finally, our induction hypothesis and the increase of f in the set S_n with respect to the indices i yields

$$\mathbf{z}^*(S_n)_f \mathbf{z} \leq k \mathbf{z}^* \mathbf{z} f(x_k) \leq k \mathbf{z}^* \mathbf{z} f(x_{k+1}). \tag{5.5}$$

Now, by combining (5.3), (5.4) and (5.5) we obtain

$$\begin{aligned}
|\mathbf{y}^*(S_n)_f \mathbf{y}| &\leq \frac{f(x_{k+1})}{2} ((k+1) |y_{k+1}|^2 + \mathbf{y}^* \mathbf{y}) \\
&\quad + \frac{f(x_{k+1})}{2} ((k-1) |y_{k+1}|^2 + \mathbf{y}^* \mathbf{y}) + k \mathbf{z}^* \mathbf{z} f(x_{k+1}) \\
&= f(x_{k+1}) \left(\mathbf{y}^* \mathbf{y} + \underbrace{k |y_{k+1}|^2 + k \mathbf{z}^* \mathbf{z}}_{=k \mathbf{y}^* \mathbf{y}} \right) = (k+1) f(x_{k+1}) \mathbf{y}^* \mathbf{y}.
\end{aligned}$$

This completes the proof. \square

Lemma 5.2. *Let $f : P \rightarrow \mathbb{R}$ be a function with nonnegative values, and let V_k denote the k -dimensional subspace of the complex vector space \mathbb{C}^n consisting of vectors that have zero entries in the coordinates at $1, 2, \dots, n-k$ (i.e. $V_k = \text{span}\{\mathbf{e}_{n-k+1}, \mathbf{e}_{n-k+2}, \dots, \mathbf{e}_n\}$). Let $\mathbf{y} = [y_1, \dots, y_n]^T$ be any vector in V_k (that is, $y_1 = \cdots = y_{n-k} = 0$). If f is order-reversing in $\text{joincl}(S_n)$, then*

$$\mathbf{y}^*[S_n]_f \mathbf{y} \leq k \mathbf{y}^* \mathbf{y} f(x_{n-k+1}). \tag{5.6}$$

Proof. The proof is very similar to the proof of Lemma 5.1 and is essentially the same as Hong and Enoch Lee's proof in [20, Theorem 2.3]. \square

By applying the Courant-Fischer theorem together with Lemmas 5.1 and 5.2 we are now able to give bounds for the eigenvalues of the matrices $(S_n)_f$ and $[S_n]_f$.

Theorem 5.1. *Let $\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_n^{(n)}$, where $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \dots \leq \lambda_n^{(n)}$, denote the eigenvalues of the matrix $(S_n)_f$. Under the assumptions of Lemma 5.1 we have*

$$\lambda_k^{(n)} \leq kf(x_k) \quad (5.7)$$

for all $k = 1, \dots, n$. Moreover, $f(x_n) \leq \lambda_n^{(n)}$.

Proof. Let $1 \leq k \leq n$. By applying Lemma 5.1 and Courant-Fischer theorem ([23, Theorem 4.2.11]) we obtain

$$\begin{aligned} kf(x_k) &\geq \max_{\mathbf{0} \neq \mathbf{y} \in W_k} \frac{\mathbf{y}^*(S_n)_f \mathbf{y}}{\mathbf{y}^* \mathbf{y}} = \max_{\mathbf{0} \neq \mathbf{y} \perp \mathbf{e}_{k+1}, \dots, \mathbf{e}_n} \frac{\mathbf{y}^*(S_n)_f \mathbf{y}}{\mathbf{y}^* \mathbf{y}} \\ &\geq \min_{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{n-k} \in \mathbb{C}^n} \left(\max_{\mathbf{0} \neq \mathbf{y} \perp \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{n-k}} \frac{\mathbf{y}^*(S_n)_f \mathbf{y}}{\mathbf{y}^* \mathbf{y}} \right) = \lambda_k^{(n)}. \end{aligned}$$

The rest of the claim follows from the Rayleigh-Ritz theorem ([23, Theorem 4.2.2]) by setting $\mathbf{y} = \mathbf{e}_n$, since

$$\lambda_n^{(n)} = \max_{\mathbf{y} \neq \mathbf{0}} \frac{\mathbf{y}^*(S_n)_f \mathbf{y}}{\mathbf{y}^* \mathbf{y}} \geq \frac{\mathbf{e}_n^*(S_n)_f \mathbf{e}_n}{\mathbf{e}_n^* \mathbf{e}_n} = f(x_n).$$

□

Theorem 5.2. *Let $\hat{\lambda}_1^{(n)}, \hat{\lambda}_2^{(n)}, \dots, \hat{\lambda}_n^{(n)}$, where $\hat{\lambda}_1^{(n)} \leq \hat{\lambda}_2^{(n)} \leq \dots \leq \hat{\lambda}_n^{(n)}$, denote the eigenvalues of the matrix $[S_n]_f$. Under the assumptions of Lemma 5.2 we have*

$$\hat{\lambda}_k^{(n)} \leq kf(x_{n-k+1}) \quad (5.8)$$

for all $k = 1, \dots, n$. In addition, $f(x_1) \leq \hat{\lambda}_n^{(n)}$.

Proof. The proof is similar to the proof of Theorem 5.1. □

Example 5.1. Let $\alpha \in \mathbb{R}^+$, $S_n = \{x_1, \dots, x_n\} \subset \mathbb{Z}^+$ and $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be the function such that $f(n) = n^\alpha$ for all $n \in \mathbb{Z}^+$. Let $(S_n)_f$ denote the power GCD matrix having $(x_i, x_j)^\alpha$ as its ij element, and let $(S_n)_f^{**}$ denote the power GCUD matrix having $((x_i, x_j)^{**})^\alpha$, the power of the greatest common unitary divisor of x_i and x_j as its ij element (d divides x_i unitarily if $d \mid x_i$ and $(d, x_i/d) = 1$). Both these matrices fulfill the assumptions of Lemma 5.1, and therefore by Theorem 5.1 $kf(x_k) = kx_k^\alpha$ is an upper bound for the k th largest eigenvalue of both $(S_n)_f$ and $(S_n)_f^{**}$. Moreover, $f(x_n) = x_n^\alpha$ is a lower bound for the largest eigenvalue of both $(S_n)_f$ and $(S_n)_f^{**}$.

Example 5.2 ([20], Theorem 2.3). Let $\alpha \in \mathbb{R}^+$, $S_n = \{x_1, \dots, x_n\} \subset \mathbb{Z}^+$ and $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be the function such that $f(n) = \frac{1}{n^\alpha}$ for all $n \in \mathbb{Z}^+$. In this case the matrix $[S_n]_f$ having $\frac{1}{[x_i, x_j]^\alpha}$ as its ij element is referred to as the reciprocal power LCM matrix of the set S_n . Let $\lambda_k^{(n)}$ denote the k th largest eigenvalue of the matrix $[S_n]_f$. Thus by Theorem 5.2 we have

$$\lambda_k^{(n)} \leq kf(x_{n-k+1}) = \frac{k}{x_{n-k+1}^\alpha}.$$

In addition, $f(x_1) = \frac{1}{x_1^\alpha} \leq \lambda_n^{(n)}$.

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